

Discreteness criterion in $SL(2, \mathbb{C})$ by a test map

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Abstract In the paper (Osaka J. Math. **46**: 403-409, 2009), Yang conjectured that a non-elementary subgroup G of $SL(2, \mathbb{C})$ containing elliptic elements is discrete if for each elliptic element $g \in G$ the group $\langle f, g \rangle$ is discrete, where $f \in SL(2, \mathbb{C})$ is a test map which is loxodromic or elliptic. The purpose of this paper is to give an affirmative answer to this question.

Keywords Discrete groups, dense groups, test map, embedding

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1 Introduction

The discreteness of Möbius groups is a fundamental problem, which has been discussed by many authors. In 1976, Jørgensen established the following discreteness criterion by using the well-known Jørgensen's inequality [9].

Theorem J. *A non-elementary subgroup G of Möbius transformations acting on $\hat{\mathbb{C}}$ is discrete if and only if for each pair of elements $f, g \in G$, the group $\langle f, g \rangle$ is discrete.*

This result shows that the discreteness of a non-elementary Möbius group depends on the information of all its rank two subgroups. The above result has been generalized by many authors by using information of partial rank two subgroups. For example, Gilman[5] and Isochenko [8] used each pair of loxodromic elements, Tukia and Wang [11] used each pair of elliptic elements.

Sullivan [10] showed that a non-elementary and non-discrete subgroup is either dense in $SL(2, \mathbb{C})$ or conjugate to a dense subgroup of $SL(2, \mathbb{R})$. This result gives an approach to studying the discreteness of Möbius groups from the topological aspect. Mainly using Sullivan's result, Yang [12] obtained some generalizations by the information of the remaining four kinds of rank two subgroups.

Recently, Chen [3] proposed to use a fixed Möbius transformation as a test map to test the discreteness of a given Möbius group. His result suggests that the discreteness is not a totally interior affair of the involved group and provides a new point of view to the discreteness problem. Yang [13] generalized some results by test maps (see Theorems 2.4-2.7) and proposed the following conjecture.

Conjecture 1.1. *Let G be a non-elementary subgroup of $SL(2, \mathbb{C})$ containing elliptic elements and f a loxodromic (resp. an elliptic) transformation. If for each elliptic element $g \in G$ the group $\langle f, g \rangle$ is discrete, then G is discrete.*

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Yang proved the above conjecture for the following two special cases (Theorems 2.9, 2.11 in [13]).

Theorem Y1. *Let G be a non-elementary subgroup of $SL(2, \mathbb{R})$ containing elliptic elements and f a loxodromic (resp. an elliptic) transformation. If for each elliptic element $g \in G$ the group $\langle f, g \rangle$ is discrete, then G is discrete.*

Theorem Y2. *Let G be a non-elementary subgroup of $SL(2, \mathbb{C})$ containing elliptic elements and f a loxodromic (resp. an elliptic) transformation with $|\text{tr}^2(f) - 4| < 1$. If for each elliptic element $g \in G$ the group $\langle f, g \rangle$ is discrete, then G is discrete.*

In $SL(2, \mathbb{R})$, since the trace is real, one can find a sequence $\{g_n\}$ of distinct elliptic elements in G such that $g_n \rightarrow I$. In fact, this is a special case (i.e. $\dim M(G) = 2$) of [4, Corollary 4.5.3]. Yang mainly used this fact to prove Theorem Y1.

While in $SL(2, \mathbb{C})$, Greenberg [7] gave an example such that G is a loxodromic group and is not discrete with $\dim M(G) = 3$. This example indicates that it is nontrivial to construct a subgroup generated by f and an elliptic element in G which is non-elementary, in which one can apply Jørgensen' inequality to obtain a contradiction.

We mention that Theorem Y2 is true but there is a gap in the proof of Theorem Y2. The author got an elliptic element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $b \neq 0 \neq c$ and

$$hgh^{-1} = \begin{pmatrix} a + c\beta & -c\beta^2 + (d - a)\beta + b \\ c & -c\beta + d \end{pmatrix}, \text{ where } h = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

Taking $\beta = \frac{d-a}{2c}$, the author mistook the second entry of hgh^{-1} for zero. In fact,

$$hgh^{-1} = \begin{pmatrix} \frac{a+d}{2} & \frac{(a+d)^2 - 4}{4c} \\ c & \frac{a+d}{2} \end{pmatrix}.$$

If the sequence $\{h_n\}$ in G converges to h , then the product $b_n c_n$ of the second and third entries of $h_n g h_n^{-1}$ converges to $\frac{(a+d)^2 - 4}{4}$ which is not zero for g being elliptic.

We can mend the the proof of Theorem Y2 as followings.

The proof of Theorem Y2. Suppose that G is not discrete.

If G is a dense subgroup in $SL(2, \mathbb{R})$ then as reasoning in Theorem Y1, we can get the result.

If G is dense in $SL(2, \mathbb{C})$, we can solve the following equation

$$-cz^2 + (d - a)z + b = 0 \tag{1}$$

to get a solution β . Construct $h = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ with this obtained β . Let $\{h_n\}$ be a sequence in G converges to h and $g_n = h_n g h_n^{-1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$. Then $b_n c_n \rightarrow 0$ and $\langle f, g_n \rangle$ is discrete and non-elementary for large n . This contradicts the Jørgensen's inequality

$$|\text{tr}(f)^2 - 4| + |\text{tr}[f, g_n] - 2| = (1 + |b_n c_n|)|r - \frac{1}{r}|^2 \geq 1.$$

The proof is complete.

Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an elliptic element and $f = \begin{pmatrix} r & 0 \\ 0 & \frac{1}{r} \end{pmatrix}$ be a loxodromic or an elliptic element. Then

$$|\text{tr}(g)^2 - 4| + |\text{tr}[g, f] - 2| = 4 - (a + d)^2 + |bc||r - \frac{1}{r}|^2. \tag{2}$$

Suppose G is dense in $SL(2, \mathbb{C})$ and non-elementary. If we can find an elliptic element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ with $4 - (a + d)^2 < 1$, then as in the above proof, we can get a sequence $\{g_n\}$ of distinct elliptic elements with $|b_n c_n| \rightarrow 0$. This may provide us a desired contradiction to prove Conjecture 1.1. Motivated by this observation, we manage to get such an elliptic element under certain condition by embedding $SL(2, \mathbb{C})$ into $U(1, 1; \mathbb{H})$.

Our main theorem is

Theorem 1.1. *Conjecture 1.1 is positive.*

2 The unitary group and embedding principle

In this section, we will recall some facts about quaternion and the quaternionic hyperbolic geometry. The reader is referred to [1, 2, 4] for more information.

Let \mathbb{H} denote the division ring of real quaternions. Elements of \mathbb{H} have the form $q = q_1 + q_2 \mathbf{i} + q_3 \mathbf{j} + q_4 \mathbf{k} \in \mathbb{H}$ where $q_i \in \mathbb{R}$ and

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1.$$

Let $\bar{q} = q_1 - q_2 \mathbf{i} - q_3 \mathbf{j} - q_4 \mathbf{k}$ be the *conjugate* of q , and

$$|q| = \sqrt{\bar{q}q} = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$$

be the *modulus* of q . We define $\Re(q) = (q + \bar{q})/2$ to be the *real part* of q , and $\Im(q) = (q - \bar{q})/2$ to be the *imaginary part* of q . Also $q^{-1} = \bar{q}|q|^{-2}$ is the *inverse* of q . We remark that for a complex number c , we have $\mathbf{j}c = \bar{c}\mathbf{j}$.

Let $\mathbb{H}^{1,1}$ be the vector space of dimension 2 over \mathbb{H} with the unitary structure defined by the Hermitian form

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J \mathbf{z} = \overline{w_1} z_1 - \overline{w_2} z_2,$$

where \mathbf{z} and \mathbf{w} are the column vectors in $\mathbb{H}^{1,1}$ with entries (z_1, z_2) and (w_1, w_2) respectively, \cdot^* denotes the conjugate transpose and J is the Hermitian matrix

$$J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We define a *unitary transformation* g to be an automorphism $\mathbb{H}^{1,1}$, that is, a linear bijection such that

$$\langle g(\mathbf{z}), g(\mathbf{w}) \rangle = \langle \mathbf{z}, \mathbf{w} \rangle \quad (3)$$

for all \mathbf{z} and \mathbf{w} in $\mathbb{H}^{1,1}$. We denote the group of all unitary transformations by $U(1, 1; \mathbb{H})$.

Following [4, Section 2], let

$$V_0 = \left\{ \mathbf{z} \in \mathbb{H}^{1,1} - \{0\} : \langle \mathbf{z}, \mathbf{z} \rangle = 0 \right\}, \quad V_- = \left\{ \mathbf{z} \in \mathbb{H}^{1,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0 \right\}.$$

It is obvious that V_0 and V_- are invariant under $U(1, 1; \mathbb{H})$. We define V^s to be $V^s = V_- \cup V_0$. Let $P : V^s \rightarrow P(V^s) \subset \mathbb{H}$ be the projection map defined by

$$P \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 z_2^{-1}.$$

We define $\mathbb{B} = P(V_-)$, the ball model of 1-dimensional quaternionic hyperbolic space. It is easy to see that \mathbb{B} can be identified with the quaternionic unit ball $\{z \in \mathbb{H} : |z| < 1\}$. Also the unit sphere in \mathbb{H} is $\partial\mathbb{B} = P(V_0)$ and the center of the ball is $0 = P\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1, 1; \mathbb{H})$ then, by definition, g preserves the Hermitian form. Hence

$$\mathbf{w}^* J \mathbf{z} = \langle \mathbf{z}, \mathbf{w} \rangle = \langle g\mathbf{z}, g\mathbf{w} \rangle = \mathbf{w}^* g^* J g \mathbf{z}$$

for all \mathbf{z} and \mathbf{w} in V . Letting \mathbf{z} and \mathbf{w} vary over a basis for V we see that $J = g^* J g$. From this we find $g^{-1} = J^{-1} g^* J$. That is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix}$$

and consequently,

$$|a| = |d|, |b| = |c|, |a|^2 - |c|^2 = 1, \bar{a}b = \bar{c}d, a\bar{c} = b\bar{d}. \quad (4)$$

As in [1, 2], we can regard $U(1, 1; \mathbb{H})$ as the isometries of real hyperbolic 4-space, whose model is the unit ball in the quaternions \mathbb{H} . $SL(2, \mathbb{C})$, the isometries of real hyperbolic 3-space, can be embedded as a subgroup of $U(1, 1; \mathbb{H})$ as following:

$$f \in SL(2, \mathbb{C}) \hookrightarrow T f T^{-1} \in U(1, 1; \mathbb{H}),$$

where

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -\mathbf{j} \\ -\mathbf{j} & 1 \end{pmatrix}.$$

Let $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$. Then

$$\hat{f} = T f T^{-1} = \frac{1}{2} \begin{pmatrix} 1 & -\mathbf{j} \\ -\mathbf{j} & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & \mathbf{j} \\ \mathbf{j} & 1 \end{pmatrix} \in U(1, 1; \mathbb{H}).$$

We mention that our model is slight different from the model in [4], where the Hermitian matrix is $J = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. It follows from (3) that both models define the same unitary group. This difference just exchanges the inner and outer of the same unit sphere of those two models.

The following lemma is crucial to us.

Lemma 2.1. (cf. [4, Corollary 4.5.2]) *Let G be a subgroup of $U(1, n; \mathbb{H})$ such that (a) G does not leave invariant a point in $\partial H_{\mathbb{H}}^n$ or a proper totally geodesic submanifold of $H_{\mathbb{H}}^n$ (b) the identity is not an accumulation point of the elliptic elements in G . Then G is discrete.*

Using the same notation as in [4], for any totally geodesic submanifold $M \in H_{\mathbb{H}}^n$, we denote by $I(M)$ the subgroup of $U(1, n; \mathbb{H})$ which leaves M invariant. By [4, Proposition 2.5.1], the proper totally geodesic submanifolds of $H_{\mathbb{H}}^1$ are equivalent to one of the four types: $H_{\mathbb{R}}^1$, $H_{\mathbb{C}}^1$ and $H^1(\mathbb{I})$.

By [4, Lemmas 4.2.1, 2], we have the following lemma.

Lemma 2.2. *Let $g \in U(1, 1; \mathbb{H})$. Then*

(i) *the elements $g \in I(H_{\mathbb{R}}^1)$ are of the form*

$$g = A\lambda, \quad A \in U(1, 1; \mathbb{R}), \lambda \in \mathbb{H}, |\lambda| = 1;$$

(ii) the elements $g \in I(H_{\mathbb{C}}^1)$ are of the form

$$g = A, A \in U(1, 1; \mathbb{C});$$

(iii) the elements $g \in I(H^1(\mathbb{I}))$ are of the form

$$g = \begin{pmatrix} a & b \\ -\varepsilon b & \varepsilon a \end{pmatrix} \in U(1, 1; \mathbb{H}), \varepsilon = \pm 1. \quad (5)$$

Lemma 2.3. *Let G be a subgroup of $SL(2, \mathbb{C})$. Then TGT^{-1} is a subgroup of $U(1, 1; \mathbb{H})$. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $TGT^{-1} \subset I(H^1(\mathbb{I}))$ then either*

- (i) $a, d \in \mathbb{R}$ and $b, c \in \mathbf{i}\mathbb{R}$, or
- (ii) $a, d \in \mathbf{i}\mathbb{R}$ and $b, c \in \mathbb{R}$.

Proof. If $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $TGT^{-1} \subset I(H^1(\mathbb{I}))$, then TgT^{-1} is of form (5). By our embedding and the fact $\mathbf{j}c = \bar{c}\mathbf{j}, \forall c \in \mathbb{C}$, we can verify that the cases $\varepsilon = 1$ and $\varepsilon = -1$ correspond to cases (i) and (ii), respectively.

Lemma 2.4. *Let G be a subgroup of $U(1, 1; \mathbb{H})$. Define $tr(g) = a + d$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. Then*

$$\Re(tr(g)) = \Re(tr(fgf^{-1})), \forall f \in U(1, 1; \mathbb{H}).$$

3 The proof of Theorem 1.1

We also need the following lemma, which is a direct consequence of the well-known proposition in [10, Section 1].

Lemma 3.1. *Let G be a non-elementary subgroup of $SL(2, \mathbb{C})$. Then either*

- (i) G is discrete, or
- (ii) G is dense in $SL(2, \mathbb{C})$, or
- (iii) G is conjugate to a dense group of $SL(2, \mathbb{R})$.

The proof of Theorem 1.1. Suppose that G is non-elementary and not discrete. If G is conjugate to a dense group of $SL(2, \mathbb{R})$ then we can obtain the result as in Theorem Y1.

In what follows, we assume that G is dense in $SL(2, \mathbb{C})$.

By our embedding, $G_1 = TGT^{-1}$ is a non-elementary and non-discrete subgroup of $U(1, 1; \mathbb{H})$. Let $M(G_1)$ be the smallest totally geodesic submanifold which is invariant under G_1 . By [4, Lemma 4.5.1], the limit set $L(G_1)$ of G_1 belongs to $\partial M(G_1)$. Normalize such that $M(G_1)$ contains 0 (abuse of notation, still denote this normalized subgroup by G_1), then $M(G_1)$ is one of the four types: $H_{\mathbb{R}}^1$, $H_{\mathbb{C}}^1$, $H^1(\mathbb{I})$ and $H_{\mathbb{H}}^1$.

Since TGT^{-1} is non-elementary, $M(G_1) \neq H_{\mathbb{R}}^1$. Suppose that $M(G_1) = H_{\mathbb{C}}^1$. By Lemma 2.2, TGT^{-1} is a subgroup of $U(1, 1; \mathbb{C})$. Since $PU(1, 1; \mathbb{C})$ is isomorphism to $PSL(2, \mathbb{R})$, we can get the result as in Theorem Y1 in this case.

Suppose that $M(G_1) = H_{\mathbb{H}}^1$. By Lemma 2.1, the identity is an accumulation point of the elliptic elements in G_1 . Therefore we get a sequence $\{g_n\}$ of distinct elliptic elements in G such that $g_n \rightarrow I$. By the same reasoning as in Theorem Y1, we can get the result.

Suppose that $M(G_1) = H^1(\mathbb{I})$. By Lemmas 2.3, 2.4, we know that the trace of $g \in G$ belongs to either \mathbb{R} or $\mathbf{i}\mathbb{R}$. Let $k \in SL(2, \mathbb{C})$ be an elliptic element with $3.1 < tr^2(k) < 3.9$. Since G is

dense in $SL(2, \mathbb{C})$, there exist a sequence k_n converges to k . Therefore we can find an elliptic element $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{C})$ with $3 < tr^2(g) < 4$. Since G is non-elementary, we can further assume that $b \neq 0 \neq c$.

Let β be a solution to the equation (1). Construct $h = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$ and let $\{h_n\}$ be a sequence in G converges to h . Then $g_n = h_n g h_n^{-1} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$ are elliptic elements with $3 < tr^2(g_n) = tr^2(g) < 4$ and $b_n c_n \rightarrow 0$ as $n \rightarrow \infty$. Note that $\langle f, g_n \rangle$ is discrete and non-elementary for large n . This contradicts the Jørgensen's inequality

$$|tr(g_n)^2 - 4| + |tr[g_n, f] - 2| = 4 - (a_n + d_n)^2 + |b_n c_n| \left| r - \frac{1}{r} \right|^2 \geq 1 \quad (6)$$

The proof is complete.

Remark. In [14], Yang asked that whether there is a non-elementary and nondiscrete subgroup of $Isom(H^3) = PSL(2, \mathbb{C})$ which contains elliptic such that each of them has order 2. By the proof of Theorem 1.1, we know that the answer to this question is *negative*.

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